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Mental And Written Computation: Abilities Of Students In A Reform-Based Curriculum

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Abstract

Two studies evaluated the computational abilities of US fifth-graders in the University of Chicago School Mathematics Projects curriculum (UCSMP), a reform-based program. In Study 1, the UCSMP students far outperformed the comparison fifth graders on a test of mental computation. On two additional fraction problems, the UCSMP fifth graders outperformed comparison seventh graders. This study replicated findings from the previous year. In Study 2, UCSMP students scored slightly higher than comparison students on a paper-and-pencil computation test, although few significant differences were found. However, major differences were found in the types of solution methods used by the two groups on the written tests. Aspects of the UCSMP curriculum, especially the focus on student discussions of multiple solution methods, are discussed in conjunction with these findings.

As part of an exercise in making up and solving number stories, a group of second graders posed the following story problem to their classmates:

"A squid is 55 feet long and a crocodile is 21 feet
How many more feet is the squid?"

In the whole-class discussion that followed, student took turns sharing a variety of mental solutions as the teacher recorded each on the overhead projector:

- 55 - 21 → 55 - 20 → 35 - 1 = 34
- 50 - 20 = 30 and 5 - 1 = 4. So 30 + 4 = 34
- 20 + 30 = 50. Plus 5 more is 35. But it was 21, so 35 - 1 = 34

A fourth solution with the student's thinking is illustrated below in more detail:

Student	:	I'm trying to think.... If you took 20 away, you would probably get out with the 30. And then since it has a one more you'd come out with 34. At first you'd have 35. And then when you minused the one......
Teacher	:	How did you get the 35?
Student	:	Because if it was 55, (Teacher records 55-20?) Because if you minused 25, then it would have to be 30....
Teacher	:	Oh I see what you're doing (writes 55 - 25 = 30)
Student	:	Cause in the ones place it's a zero. So you have to minus it as a 5.
Teacher	:	Have I done this the way you want? (indicating equation)
Student	:	Yes
Teacher	:	Now where are you going to get the other 4 from?
Student	:	Then since you minused the 25, then you have to plus the 4 more. (Teacher adds 55 - 25 = 30 + 4 = 34)

Several things are noteworthy about this exchange. While traditional mathematics instruction in the United States and other nations has generally focused on the practice of one standard written algorithm for each of the arithmetic operations, here four different methods were presented by the students. Rather than relying on pencil and paper, these second graders employed a variety of mental methods, including techniques that are indicative of good mental calculators, such as reformulation, 55 - 21 = (55 - 20) - 1, or subtracting tens and ones separately (Hope & Sherrill, 1987). The complexity of thinking and number knowledge is illustrated in the last solution, where the student seemed to use two correct methods, starting with:

$$55 - 21 = (55 - 20) - 1$$

and then, when asked to explain further, switching to:

$$(55 - 25) + 4$$

A second difference is that in sharp contrast to traditional instruction, where a single algorithm is practiced on a number of problems, a considerable amount of time was spent describing various solution methods on one problem. Finally, and perhaps most importantly, students actively took part both in constructing a problem and in explaining fairly complex solution methods. In probing her students' thinking and recording **their** methods on the overhead, the teacher took quite a different role from those in more traditional primary mathematics classes (e.g., Stodolsky, 1988). Following the students' mental solutions might no doubt have proved a challenge to the teacher.

The scenario described above is somewhat typical of classes in the US which have adopted reform mathematics curriculum. In particular, the curriculum investigated in this report is the University of Chicago School Mathematics Projects' (UCSMP) elementary curriculum, *Everyday Mathematics*, one of the curricula that reflect the reform movement in the US (NCTM, 1989, 1991). Some aspects of this program are:

- Students are not taught specific algorithms during the primary grades. Instead they are encouraged to develop methods that fit particular problems.
- Following exploratory exercises that build conceptual meanings for multiplication and division, students are taught non-standard algorithms for multiplication and division. For multiplication, these include the lattice and the partial products methods. Student invented methods, such as repeated addition, are still allowed and encouraged.
- Concepts and skills are generally presented in a context and practiced in games.
- Calculators, manipulatives, representations (e.g., hundreds charts), and mathematical tools (e.g., rulers and geometric templates) are regularly used during activities.
- Geometry, data and graphing, algebraic ideas, and estimation are regular strands.

As noted above, one of the hallmarks of the UCSMP curriculum is that students are encouraged to invent their own algorithms. In doing so, it is hoped that a more flexible number sense will develop (Kamii, Lewis & Livingston, 1993; Sowder, 1992). For example; when presented with a problem like 701 - 2, students with a good number sense would see that using the standard subtraction algorithm with borrowing is unnecessary. One reasonable approach might be to say, "701 take away one is 700. And if I take away one more, it's 699, " or some similar use of number knowledge. Similarly, we might hope that when solving 700 * 30, students would not use paper-and-pencil or a calculator, but instead employ basic facts and knowledge of powers of tens to solve the problem mentally.

While class observations have indicated success in this area, a number of important questions remain. One reason that standard written algorithms have been traditionally taught is because of their generalizability. If students are left to discover and use their own methods, or use non-standard methods, how efficient and generalizable will these methods be? For example, while adding up is a reasonable way to solve 45 - 39 (i.e., 39 + 1 = 40; 40 + 5 = 45. So "6" is the difference), this method would be less efficient in solving 1003 - 257. Secondly, students might become quite adept at mentally solving easy problems and relying on calculators for more difficult problems, thereby failing to develop and practice written methods for mid-size numbers. While some educators argue that pencil-and-paper skills are unnecessary in a calculator age (Wheatley & Shumway, 1992), most agree that pencil-and-paper algorithms are still necessary, although not necessarily the ones traditionally taught. Thirdly, to what degree do students in a reform-based program like the UCSMP curriculum develop number sense and facility with mental computations? While it seems reasonable to expect that an emphasis on conceptual understanding during problem solving will facilitate number sense, others might argue that automaticity in basic skills will not develop. In fact, this is one of the criticisms of the current reform movement (Willoughby, 1996). Furthermore, because more time is spent on geometry, data, and problem-solving activities, less time is available for practicing computation.

The purpose of these two studies was to investigate the computational skills of students in the UCSMP curriculum on a mental computation test as well as a more traditional paper-and-pencil test. These are of interest for different reasons. In the case of mental computation, past research has shown the poor performance of students on mental computation (Reys & Barger, 1994). One reason might be that mental computation requires flexibility in choosing solution methods, a facility with manipulating numbers mentally, and some sensitivity to the reasonableness of an answer - all components of number sense which are not addressed during practice of standard algorithms. A test of mental computation seems pertinent to evaluating the success of the UCSMP curriculum in helping students develop number sense.

Proficiency on more standard pencil-and-paper tasks is of interest for a different reason. Reform-based curricula are not likely to succeed if scores drop on standardized tests, and these tests focus heavily on computation. This issue was investigated in Study 2. These two studies then investigated the success of the UCSMP curriculum in developing number sense without losing ground in more traditional areas.

STUDY 1

In a recent study, Reys, Reys and Hope (1993) tested students in two districts for the purpose of providing a baseline in mental computation. Reys' sample consisted of second, fifth, and seventh graders in traditional instruction, and results were generally poor. During the field test of *Fifth Grade Everyday Mathematics* (Bell et al, 1995), students in the UCSMP curriculum outperformed the Reys' sample on nearly all questions when administered the same test (Carroll, 1996). The purpose of the current study was to replicate the earlier research with students using the revised UCSMP fifth-grade materials.

Participants

Fifth-graders in four classes using the revised UCSMP curriculum were selected (n = 84 students). The demographics, three suburban and one urban school, were similar to those in Reys et al. study (1993). All classes were of average ability, with higher-achievers being pulled out in three of the classes, and all students had been in the UCSMP curriculum since kindergarten, with the exception of transfer-ins. All four teachers had taken part in the field test of *Fifth Grade Everyday Mathematics* the previous year.

Procedure

As in the original study, students were tested in a whole-class format. Students were instructed to solve all problems mentally and were provided with a narrow sheet of paper for recording only their answers. The test consisted of 26 fifth-grade items and two seventh-grade items from the Reys' test. The two seventh-grade items were chosen to include some fraction problems. Eleven of the questions were presented orally, 13 (including the two fraction problems) were presented visually on the overhead, and four story problems were presented visually and read aloud. Oral problems were repeated once, and the time of 10 seconds was allowed on each problem. All problems were suited for mental computation. Tests were administered by the author.

Results and discussion

UCSMP students outperformed the comparison fifth graders on nearly all problems. Eighteen of these differences were significant (Chi-square test, $df = 1$, $p < .05$), with all but one significant difference (8 * 99) favoring the UCSMP group.

Many of the differences favoring the UCSMP group were quite large. Table 1 shows the problems in the order given. UCSMP students scored higher than the comparison students on all three subtests.

Table 1: Percent Correct on Mental Computation Test

Question	*Everyday Mathematics*	Baseline Group (Reys et. al, 1993)
ORAL PRESENTATION		
1. 47 + 29	41%	35%
2. 28 + 75	40%	34%
3. 265 - 98	11%	6%
4. Double 84	70%	50% *
5. 60 × 70	61%	33% *
6. 4000 × 100	47%	17% *
7. 8 × 99	11%	26% *
8. 5 × 125	38%	15% *
9. 6 × 55	32%	33%
10. 5 × 54	32%	20% *
11. 3800 ÷ 10	24%	12% *
ORAL TOTAL	37%	26%
VISUAL PRESENTATION		
1. 68 + 32	85%	53% *
2. 470 - 300	57%	64%
3. 325 + 25 + 75	63%	39% *
4. 75 + 85 + 25 + 2000	43%	1% *
5. 426 + 75	51%	37% *
6. $20.00 - $11.98	13%	10%
7. 7000 - 4000 - 300	36%	18% *
8. 25 * 28	4%	1%
9. 2 * 27 * 5	23%	12% *
10. 3500 ÷ 35	42%	16% *
11. 8 * 1.99	13%	5% *
VISUAL TOTAL	39%	11%

Table 1 (cont'd)

Question	*Everyday Mathematics*	Baseline Group (Reys et. al, 1993)
WORD PROBLEMS		
1. Kevin delivers 38 newspapers each day. How many does he deliver in 5 days?	35%	26%
2. Five identical tapes cost $10.30. What does each tape cost?	12%	4% *
3. Linda had $20.00. How much will she have left if she buys this scarf? ($12.85)	18%	5% *
4. A large arena contains 2000 seats. If there are 50 equal rows, how many seats are in each row?	18%	5% *
WORD PROBLEM TOTAL	21%	11%
TOTAL ON ALL 26 PROBLEMS	35%	22%

Note: * denotes significance at the .05 level on Chi-square test.

Certain patterns are apparent in the results. For example, on all multiplication and division questions involving powers of tens, UCSMP students did especially well. On these four questions, UCSMP students had a mean correct score of 44% compared to 20% for the comparison group. Similarly, on sums and differences of more than two numbers (e.g., 325 + 25 + 75), UCSMP students had a mean correct score of 48% compared to 19% for the comparison group. UCSMP students generally scored two to three times higher on the word problems.

Certain areas of weakness stand out also. For example, only about 10% of the UCSMP students successfully worked with computations involving nines such as 8 * 99 or 8 * $1.99. Similarly, only about 10% successfully subtracted 265 - 98, although this was a higher proportion than the comparison students. Apparently, few of the UCSMP students recognized that these problems could be simplified using knowledge of tens and hundreds, e.g., 8 * 100 - 8 or 265 - 100 + 2. Of course, the oral presentation, which would have limited memory, or the time limit might have been factors.

Individual interviews with fourth, fifth, and sixth graders, carried on as part of field test, suggest reasons for these strengths and relative weaknesses. UCSMP students continue to use the mental methods that they developed during the primary grades to solve problems. These involve the use of relationships between operations, e.g., switching subtraction, as well as addition left-to-right or reformulation methods. While these work well for some problems, e.g., 7000 - 4000 - 300, they are not the most efficient on others. For example, many fifth-graders interviewed the previous years solved 265 - 98 as 265 - 90 - 8 which led to errors. Thus, while the results are largely positive, they also suggest that UCSMP might benefit from more instruction in specific strategies or mental "tricks". Performance on multiplication and division of powers of ten, a topic in which students do have frequent practice during fourth grade, was quite high.

Finally, on the two fraction questions, UCSMP fifth graders far outperformed the comparison seventh graders. The question 4 - 1½, was answered correctly by 49% of the UCSMP fifth graders and 27% of the comparison seventh graders. The question "What is $\frac{5}{8}$ of 40?" was answered correctly by 39% of the UCSMP and only 6% of the comparison group. Because UCSMP students begin conceptual work with fractions in first grade and practice mental computations ("What is ½ of 8?") throughout the primary grades, this is not unexpected. Importantly, it provides evidence for the strength of the UCSMP curriculum in areas on which US students typically show poor understanding.

Results then replicate the study done the previous year with nearly identical findings: UCSMP students are much better at mental computation than students in traditional instruction. Furthermore, this ability extends to work with fractions. This suggests that in many ways, UCSMP's focus on invented algorithms, student discussions of multiple solutions, and other aspects of number sense is largely successful. Results also suggest that teaching of a few efficient strategies, or encouraging students' discoveries and discussions of methods, might improve performance. Study 2 examines the performance of UCSMP students on another area of computation - more standard paper-and-pencil computations.

STUDY 2

Participants

Four UCSMP fifth grade classes (n = 88) were selected for this study. Two classes were in suburban districts and two were in rural / small town schools. The two suburban classes had taken part in Study 1. Except for transfer-ins, all students had been in the UCSMP curriculum since kindergarten, and teachers had used the field test materials the previous year. Four comparison classes (n = 89) in two districts were also selected to match the UCSMP classes on demographics and location, i.e. two suburban and two small town / rural schools. Students in the comparison classes used more traditional basal texts. Classes were paired based on similar demographics.

Procedure

Tests were administered to the whole class by classroom teachers in early February and returned to UCSMP for scoring and analysis. Students were told they could solve the problems however they wished, although calculators or other tools were not available. Classes were allowed up to 30 minutes to complete the test, which consisted of 14 symbolic computation items - three addition, three subtraction, five multiplication, and three division. Students were also asked to indicate whether they solved one problem "10.15 ÷ 5", mentally, using pencil-and-paper, or by another method. Three problems involved decimals and one involved fraction multiplication.

Results

Overall scores indicated no significant difference between the UCSMP and comparison groups with both answering about 75% of the questions correctly (see Table 2). Two class differences favored the UCSMP students and two favored the comparison students, but only one difference was significant , $T(44) = 2.16$, $p < .05$, favoring a UCSMP class.

Table 2. Scores on Written Computation Test

Pair	*Everyday Mathematics*	Comparison
1	72%	83%
2	77%	68%
3	72%	76%
4	83%	69% *
TOTAL	76%	74%

Note: * denotes significance on t-test at the .05 level.

Across the 15 problems, there were relatively few differences between groups (see Table 3). Chi-square tests on individual found only three significant differences ($p < .05$) between the groups, two favoring the UCSMP group and one favoring the comparison students. As in the mental computation test, UCSMP students scored higher on problems involving multiplication and division of powers of ten as well as finding ¼ of 28. Both groups tended to score higher on addition and multiplication than on subtraction or division, although UCSMP students scored higher when a strategy could be used, e.g., 360 ÷ 6 = 10 * (36 ÷ 6), or 2 * 37 * 5 = (2 * 5) * 37.

Table 3. Percent Correct on Individual Written Computation Questions

Question	*Everyday Mathematics* (n = 88)	Comparison (n = 89)
1. 25 + 99 + 5	93%	97%
2. 128 + 275	91%	87%
3. 20 - 11.98	67%	72%
4. 265 - 98	81%	83%
5. 99 * 8	83%	83%
6. 28 * 25	74%	77%
7. 2.28 + 1.75	84%	79%
8. 360 ÷ 6	80%	66% *
9. ¼ of 28	88%	52% *
10. 2 * 37 * 5	75%	66%
11. 700 * 30	85%	75%
12. 110 ÷ 5	73%	74%
13. 3001 - 9	75%	75%
14. 9018 ÷ 6	40%	55% *
15. 10.15 ÷ 5	57%	66%
TOTAL	76%	74%

Note: * denotes significant difference between groups, Chi-square test, $p < .05$.

<u>Solution procedures</u> While this was a paper-and-pencil test, it might be hoped that UCSMP students use alternative methods when they better suit the problem. Two measures of alternative methods were used. First, students were asked directly to indicate how they solved "10.15 ÷ 5." For this problem, 50% of the UCSMP students reported that they used pencil and paper while 47% reported using a mental solution. In contrast, 75% of the comparison students reported using pencil-and-paper and only 18% used a mental method. Thus, nearly three

times as many UCSMP students as comparison students reported approaching this problem mentally.

Second, test papers were examined for the solution methods used. Although solution methods were not always obvious, it was fairly clear when a student used a standard written algorithm - showing the "borrowing" or "carrying" - or an alternate method like lattice multiplication. In some cases an incorrect procedure was obvious, such as adding for subtraction or multiplication. In some cases, the method was not shown or obvious, and these were recorded as such. Table 4 shows the results on the three multiplication problems.

Several patterns are apparent in the table. First, UCSMP students were much more likely to use an alternative algorithm. For example, about a third used the lattice method on two of the problems. Second, results suggest that UCSMP students were more likely to use a mental solution on 700 * 30. That is, 59% of the UCSMP students showed no written method on this problem compared to 35% of the comparison students. Finally, the error rates across the problems were fairly similar for both groups. Interestingly, although UCSMP students were not taught the standard multiplication algorithm, their error rate was sometimes lower than that of the comparison students on this method. Not surprisingly, none of the comparison students used the lattice method, and fewer used other alternative methods. In short, UCSMP students showed a greater variety of solution methods across the problems and showed a better understanding of when to apply different methods, e.g., when solving 700 * 30. Similar results were found for the three subtraction problems.

Table 4. Percent using Solution Methods for Whole-number Multiplication (and Percent Correct)

Problem	Standard written algorithm	Lattice method	Method not shown	Other method or algorithm	Faulty procedure or not done
Everyday Mathematics					
8 * 99	30% (9%)	31% (7%)	28% (27%)	3% (33%)	2% (100%)
28 * 25	41% (22%)	38% (18%)	13% (36%)	6% (20%)	3% (100%)
700 * 30	17% (7%)	18% (19%)	59% (8%)	3% (67%)	2% (100%)

Table 4 (cont'd)

Problem	Standard written algorithm	Lattice method	Method not shown	Other method or algorithm	Faulty procedure or not done
Comparison					
8 * 99	75% (13%)	none	25% (32%)	none	none
28 * 25	90% (18%)	none	1% (100%)	none	9% (100%)
700 * 30	54% (15%)	none	35% (29%)	11% (40%)	1% (100%)

Finally, a comparison of Tables 1 and 3 shows that eight of the problems in Study 2 were also given on the mental computation test. Results show that UCSMP students scored much higher on the written test. However, considering the time differential, ten seconds per problem on the mental computation test compared to thirty minutes on the written test, this is not surprising.

Summary

The results of these two studies paint an optimistic picture of the effects of a reform-based curriculum. UCSMP fifth graders scored much higher on a mental computation test and about the same as a comparison group on a more standard pencil-and-paper assessment. One conclusion is that the UCSMP curriculum has helped students to develop a better number sense - enabling them to apply efficient strategies like left-to-right addition and reformulation of problems. Links between operations, e.g., solving subtraction by adding up or multiplication by repeated additions, allow greater flexibility in solving a problem and build important connections between mathematical ideas. The classroom vignette given at the beginning of this paper suggests how this is encouraged and developed in the UCSMP curriculum. Rather than merely looking for "the correct answer", the teacher elicited the students' processes, probed their reasoning, sought multiple solutions, and modeled these methods on the overhead. Instead of using one solution that fits all problems for a given operation, students in this curriculum seek solution methods that best fit a particular problem. In short, understanding rather than algorithmic behavior is encouraged.

UCSMP students also applied some alternative methods on the paper-and-pencil test in Study 2. For example, more UCSMP students indicated that they solved a division problem mentally, and more uses of different multiplication algorithms were apparent. UCSMP students scored slightly higher on the written computation test. Given that much time was spent on geometry, data, and problem solving activities, these computation results are quite favorable.

While these studies indicate that UCSMP students have not lost ground in more traditional bare computation, other research studies have found considerable strengths in other areas. For example, on an end-of-year geometry test given to the classes from Study 2, mean correct scores were 61% for the UCSMP group and 41% for the comparison group. Similar results were found on a geometric reasoning subtest.

One question that remains unresolved is the relative benefits of allowing students to invent their own methods as opposed to providing some direct instruction in mental strategies. On one hand, UCSMP students did not always apply the most efficient mental strategies, e.g., on 265 - 98. Individual interviews confirmed this. However, the strength of their number sense is probably due to allowing them to invent their own methods. The degree to which these two approaches can be melded is an issue for future research.

A second question is whether the results, especially on the mental computation test, actually indicate a better number sense, or merely, better skill at mental computation. Surely, understanding of number and operations relationships as well as some mathematical properties (e.g., commutative properties) are indicated. But other aspects of number sense, such as estimation or place value, were not assessed. The links between reform-based instruction and the multiple aspects of number sense should be investigated in further research.

Finally, while some UCSMP students used their understanding of alternative methods on the paper-and-pencil test, others reverted to standard algorithms, even on problems like 3001 - 9 or 20 - 11.98. Research in how to help students access their number sense, i.e., how to help students develop and use better metacognitive skills in various situations, is needed.

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